

NEW FORMULAS FOR THE SPECTRAL RADIUS VIA ALUTHGE TRANSFORM

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ABSTRACT. In this paper we give several expressions of spectral radius of a bounded operator on a Hilbert space, in terms of iterates of Aluthge transformation, numerical radius and the asymptotic behavior of the powers of this operator.

Also we obtain several characterizations of normaloid operators.

1. INTRODUCTION

Let H be complex Hilbert spaces and $\mathcal{B}(H)$ be the Banach space of all bounded linear operators from H into it self.

For $T \in \mathcal{B}(H)$, the spectrum of T is denoted by $\sigma(T)$ and $r(T)$ its spectral radius. We denote also by $W(T)$ and $w(T)$ the numerical range and the numerical radius of T .

As usually, for $T \in \mathcal{B}(H)$ we denote the module of T by $|T| = (T^*T)^{1/2}$ and we shall always write, without further mention, $T = U|T|$ to be the unique polar decomposition of T , where U is the appropriate partial isometry satisfying $\mathcal{N}(U) = \mathcal{N}(T)$. The Aluthge transform introduced in [1] as

$$\Delta(T) = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}, \quad T \in \mathcal{B}(H),$$

to extend some properties of hyponormal operators. Later, in [8], Okubo introduced a more general notion called λ -Aluthge transform which has also been studied in detail.

For $\lambda \in [0, 1]$, the λ -Aluthge transform is defined by,

$$\Delta_\lambda(T) = |T|^\lambda U |T|^{1-\lambda}, \quad T \in \mathcal{B}(H).$$

Notice that $\Delta_0(T) = U|T| = T$, and $\Delta_1(T) = |T|U$ which is known as Duggal's transform. It has since been studied in many different contexts and considered by a number of authors (see for instance, [1, 2, 3, 7, 6, 5] and some of the references there). The interest of the Aluthge transform lies in the fact that it respects many properties of the original operator. For example, (see [5, Theorems 1.3, 1.5])

$$(1.1) \quad \sigma(\Delta_\lambda(T)) = \sigma(T), \text{ for every } T \in \mathcal{B}(H),$$

Another important property is that $\text{Lat}(T)$, the lattice of T -invariant subspaces of H , is nontrivial if and only if $\text{Lat}(\Delta(T))$ is nontrivial (see [5, Theorem 1.15]).

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Moreover, Yamazaki ([12]) (see also, [11, 10]), established the following interesting formula for the spectral radius

$$(1.2) \quad \lim_{n \rightarrow \infty} \|\Delta_\lambda^n(T)\| = r(T)$$

where Δ_λ^n the n -th iterate of Δ_λ , i.e $\Delta_\lambda^{n+1}(T) = \Delta_\lambda(\Delta_\lambda^n(T))$, $\Delta_\lambda^0(T) = T$.

In this paper we give several expressions of the spectral radius of an operator. Firstly in terms of the Aluthge transformation (section 2), and secondly, in section 3, we give several expressions of spectral radius, based on numerical radius and Aluthge transformation. Also, We infer several characterizations of normaloid operators (i.e. $r(T) = \|T\|$).

2. FORMULAS OF SPECTRAL RADIUS VIA ALUTHGE TRANSFORM

In this section, we use the of Rota's Theorem, order to obtain new formulas of spectral radius via Aluthge transformation

Theorem 2.1. For every operator $T \in \mathcal{B}(H)$, we have

$$\begin{aligned} r(T) &= \inf\{\|\Delta_\lambda(XTX^{-1})\|, X \in \mathcal{B}(H) \text{ invertible} \} \\ &= \inf\{\|\Delta_\lambda(e^A T e^{-A})\|, A \in \mathcal{B}(H) \text{ self adjoint} \}. \end{aligned}$$

Proof. For every invertible operator $X \in \mathcal{B}(H)$, by (1.1), we have

$$\sigma(\Delta_\lambda(XTX^{-1})) = \sigma(XTX^{-1}) = \sigma(T).$$

It follows that

$$r(T) = r(\Delta_\lambda(XTX^{-1})) \leq \|\Delta_\lambda(XTX^{-1})\| \text{ for every invertible operator } X \in \mathcal{B}(H).$$

Hence

$$\begin{aligned} r(T) &\leq \inf\{\|\Delta_\lambda(XTX^{-1})\|; X \in \mathcal{B}(H) \text{ invertible} \} \\ &\leq \inf\{\|\Delta_\lambda(\exp(A)T \exp(-A))\|; A \in \mathcal{B}(H) \text{ self adjoint} \}, \end{aligned}$$

In the other hand, for $\varepsilon > 0$, we have

$$r\left(\frac{T}{r(T) + \varepsilon}\right) = \frac{r(T)}{r(T) + \varepsilon} < 1.$$

From Rota's Theorem [9, Theorem 2], $\frac{T}{r(T) + \varepsilon}$ is similar to an contraction. Thus there exists an invertible operator $X_\varepsilon \in \mathcal{B}(H)$ such that

$$(2.1) \quad \|\Delta_\lambda(X_\varepsilon T X_\varepsilon^{-1})\| \leq \|X_\varepsilon T X_\varepsilon^{-1}\| \leq r(T) + \varepsilon.$$

Now, let $X_\varepsilon = U_\varepsilon |X_\varepsilon|$ be the polar decomposition of X_ε . Clearly U_ε is a unitary operator, and $|X_\varepsilon|$ is invertible. Therefore there exist $\alpha > 0$ such that $\sigma(|X_\varepsilon|) \subseteq [\alpha, +\infty[$. Consequently $A_\varepsilon = \ln(|X_\varepsilon|)$ exists and it is self adjoint, we also have

$$|X_\varepsilon| = e^{A_\varepsilon} \text{ and } |X_\varepsilon|^{-1} = e^{-A_\varepsilon}.$$

Thus

$$\begin{aligned}\|\Delta_\lambda(X_\varepsilon T X_\varepsilon^{-1})\| &= \|\Delta_\lambda(U e^{A_\varepsilon} T e^{-A_\varepsilon} U^*)\| \\ &= \|U \Delta_\lambda(e^{A_\varepsilon} T e^{-A_\varepsilon}) U^*\| \\ &= \|\Delta_\lambda(e^{A_\varepsilon} T e^{-A_\varepsilon})\|.\end{aligned}$$

Hence

$$\|\Delta_\lambda(e^{A_\varepsilon} T e^{-A_\varepsilon})\| \leq \|X_\varepsilon T X_\varepsilon^{-1}\| \leq r(T) + \varepsilon.$$

It follows that for all $\varepsilon > 0$,

$$\begin{aligned}r(T) &\leq \inf\{\|\Delta_\lambda(X T X^{-1})\|, X \in \mathcal{B}(H) \text{ invertible} \} \\ &\leq \inf\{\|\Delta_\lambda(e^A T e^{-A})\|, A \in \mathcal{B}(H) \text{ self adjoint} \} \\ &\leq \|\Delta_\lambda(e^{A_\varepsilon} T e^{-A_\varepsilon})\| \leq \|X_\varepsilon T X_\varepsilon^{-1}\| \\ &\leq r(T) + \varepsilon.\end{aligned}$$

Finally, since $\varepsilon > 0$ is arbitrary, we obtain

$$\begin{aligned}r(T) &= \inf\{\|\Delta_\lambda(X T X^{-1})\|, X \in \mathcal{B}(H) \text{ invertible} \} \\ &= \inf\{\|\Delta_\lambda(e^A T e^{-A})\|, A \in \mathcal{B}(H) \text{ self adjoint} \}.\end{aligned}$$

Theroforte the proof of Theorem is complete. \square

As immediate consequence of the Theorem 2.1, we obtain the following corollary which gives a formula of the spectral radius based on n -th iterate of Δ_λ .

Corollary 2.1. If $T \in \mathcal{B}(H)$, then for every $n \geq 0$,

$$\begin{aligned}r(T) &= \inf\{\|\Delta_\lambda^n(X T X^{-1})\|, X \in \mathcal{B}(H) \text{ invertible} \} \\ &= \inf\{\|\Delta_\lambda^n(e^A T e^{-A})\|, A \in \mathcal{B}(H) \text{ self adjoint} \}.\end{aligned}$$

Proof. First, note that $\|\Delta_\lambda(T)\| \leq \|T\|$, consequently we have

$$(2.2) \quad \|\Delta_\lambda^n(T)\| \leq \|\Delta_\lambda^{n-1}(T)\| \leq \dots \leq \|\Delta_\lambda(T)\| \leq \|T\|, \quad \forall n \in \mathbb{N}^*.$$

Now, clearly $\sigma(\Delta_\lambda^n(T)) = \sigma(T)$, for all $n \in \mathbb{N}$. It follows that, for every invertible operator $X \in \mathcal{B}(H)$ we have

$$\begin{aligned}r(T) &= r(\Delta_\lambda^n(X T X^{-1})) \\ &\leq \|\Delta_\lambda^n(X T X^{-1})\| \\ &\leq \|\Delta_\lambda(X T X^{-1})\|.\end{aligned}$$

Therefore

$$\begin{aligned}r(T) &\leq \inf\{\|\Delta_\lambda^n(X T X^{-1})\|, X \in \mathcal{B}(H) \text{ invertible} \} \\ &\leq \inf\{\|\Delta_\lambda^n(e^A T e^{-A})\|, A \in \mathcal{B}(H) \text{ self adjoint} \} \\ &\leq \inf\{\|\Delta_\lambda(e^A T e^{-A})\|, A \in \mathcal{B}(H) \text{ self adjoint} \} \\ &= r(T).\end{aligned}$$

\square

An operator T is said to be normaloid if $r(T) = \|T\|$.

As immediate consequence of the Corollary 2.1, we obtain the following corollary which is a characterization of normaloid operators via λ -Aluthge transformation :

Corollary 2.2. If $T \in \mathcal{B}(H)$, then the following assertions are equivalent

- (i) T is normaloid;
- (ii) $\|T\| \leq \|\Delta_\lambda(XTX^{-1})\|$, for all invertible $X \in \mathcal{B}(H)$;
- (iii) $\|T\| \leq \|\Delta_\lambda^n(XTX^{-1})\|$, for all invertible $X \in \mathcal{B}(H)$ and for all natural number n .

As immediate consequence of the Corollary 2.2, we obtain a new characterization of normaloid operators

Corollary 2.3. If $T \in \mathcal{B}(H)$, then the following assertions are equivalent

- (i) T is normaloid;
- (ii) $\|T\| \leq \|XTX^{-1}\|$, for all invertible $X \in \mathcal{B}(H)$;

Theorem 2.2. Let $T \in \mathcal{B}(H)$. Then for each natural number n , we have

$$\begin{aligned} r(T) &= \lim_k \|\Delta_\lambda^n(T^k)\|^{1/k} \\ &= \lim_k \|\Delta_\lambda(T^k)\|^{1/k}. \end{aligned}$$

Proof. Note that,

$$(2.3) \quad r(T) = r(\Delta_\lambda^n(T)) \leq \|\Delta_\lambda^n(T)\| \leq \|\Delta_\lambda(T)\| \leq \|T\| \quad \forall n \in \mathbb{N}^*.$$

Let $k \in \mathbb{N}$ be arbitrary, we have

$$r(T)^k = r(T^k) = r(\Delta_\lambda^n(T^k)) \leq \|\Delta_\lambda^n(T^k)\| \leq \|\Delta_\lambda(T^k)\| \leq \|T^k\| \quad \forall n \in \mathbb{N}.$$

Hence

$$r(T) \leq \|\Delta_\lambda^n(T^k)\|^{1/k} \leq \|\Delta_\lambda(T^k)\|^{1/k} \leq \|T^k\|^{1/k}.$$

Therefore

$$r(T) \leq \lim_k \|\Delta_\lambda^n(T^k)\|^{1/k} \leq \lim_k \|\Delta_\lambda(T^k)\|^{1/k} \leq \lim_k \|T^k\|^{1/k} = r(T).$$

Which completes the proof. □

As immediate consequence of Theorem 2.2, we obtain the following corollary which is a new characterization of normaloid operators

Corollary 2.4. If $T \in \mathcal{B}(H)$, then the following assertions are equivalent

- (i) T is normaloid;
- (ii) $\|T\|^k = \|\Delta_\lambda(T^k)\|$, for all natural number k ;
- (iii) $\|T\|^k = \|\Delta_\lambda^n(T^k)\|$, for every natural number k, n

3. SPECTRAL RADIUS VIA NUMERICAL RADIUS AND ALUTHGE TRANSFORM

For $T \in \mathcal{B}(H)$, we denote the numerical range and numerical radius of T by $W(T)$ and $w(T)$, respectively.

$$W(T) = \{ \langle Tx, x \rangle; \|x\| = 1 \} \quad \text{and} \quad w(T) = \sup\{|\lambda|; \lambda \in W(T)\}.$$

In the following theorem, we obtain a new expression of the spectral radius by means of the numerical radius and Aluthge transform.

Theorem 3.1. For every operator $T \in \mathcal{B}(H)$ and for each natural number n , we have

$$\begin{aligned} r(T) &= \inf\{w(\Delta_\lambda^n(XTX^{-1})), X \in \mathcal{B}(H) \text{ invertible} \} \\ &= \inf\{w(\Delta_\lambda^n(e^A T e^{-A})), A \in \mathcal{B}(H) \text{ self adjoint} \}. \end{aligned}$$

Proof. It is well known that $r(T) \leq w(T) \leq \|T\|$. Thus, for all $X \in \mathcal{B}(H)$, invertible and for each natural number n , we have

$$r(T) = r(\Delta_\lambda^n(XTX^{-1})) \leq w(\Delta_\lambda^n(XTX^{-1})) \leq \|\Delta_\lambda^n(XTX^{-1})\|$$

It follows that

$$\begin{aligned} r(T) &\leq \inf\{w(\Delta_\lambda^n(XTX^{-1})); X \in \mathcal{B}(H) \text{ invertible} \} \\ &\leq \inf\{w(\Delta_\lambda^n(\exp(A)T \exp(-A))); A \in \mathcal{B}(H) \text{ self adjoint} \}, \\ &\leq \inf\{\|\Delta_\lambda^n(\exp(A)T \exp(-A))\|; A \in \mathcal{B}(H) \text{ self adjoint} \} \\ &= r(T) \quad (\text{by Corollary 2.1}). \end{aligned}$$

Hence we obtain the desired equalities. \square

For a bounded linear operator S , we will write $\mathcal{R}e(S) = \frac{1}{2}(S + S^*)$, the real part of S . And we denote by $\overline{W}(S)$ the closure of the numerical range of S . Then we have the following result

Theorem 3.2. For every operator $T \in \mathcal{B}(H)$, there exists $\theta \in \mathbb{R}$ such that for all natural number n ,

$$\begin{aligned} r(T) &= \inf\{w(\mathcal{R}e(\Delta_\lambda^n(\exp(i\theta)XTX^{-1}))), X \in \mathcal{B}(H) \text{ invertible} \} \\ &= \inf\{\|\mathcal{R}e(\Delta_\lambda^n(\exp(i\theta)XTX^{-1}))\|, X \in \mathcal{B}(H) \text{ invertible} \}. \end{aligned}$$

Proof. First assume that $r(T) \in \sigma(T)$. Then for all invertible $X \in \mathcal{B}(H)$, we have

$$r(T) \in \mathcal{R}e(\sigma(T)) = \mathcal{R}e(\sigma(\Delta_\lambda^n(XTX^{-1}))).$$

Thus

$$r(T) \in \mathcal{R}e(\sigma(\Delta_\lambda^n(XTX^{-1}))) \subseteq \mathcal{R}e(\overline{W}(\Delta_\lambda^n(XTX^{-1}))) = \overline{W}(\mathcal{R}e(\Delta_\lambda^n(XTX^{-1})))$$

which implies

$$\begin{aligned} r(T) &\leq w(\mathcal{R}e(\Delta_\lambda^n(XTX^{-1}))) \\ &\leq \|\mathcal{R}e(\Delta_\lambda^n(XTX^{-1}))\| \\ &\leq \|\Delta_\lambda^n(XTX^{-1})\|. \end{aligned}$$

Since the last inequalities are satisfied for all $X \in \mathcal{B}(H)$ invertible, we obtain

$$\begin{aligned} r(T) &\leq \inf\{w(\mathcal{R}e(\Delta_\lambda^n(XTX^{-1}))) \mid X \in \mathcal{B}(H) \text{ invertible} \} \\ &\leq \inf\{\|\mathcal{R}e(\Delta_\lambda^n(XTX^{-1}))\| \mid X \in \mathcal{B}(H) \text{ invertible} \} \\ &\leq \inf\{\|\Delta_\lambda^n(XTX^{-1})\| \mid X \in \mathcal{B}(H) \text{ invertible} \} \\ &= r(T) \quad (\text{by Corollary 2.1}). \end{aligned}$$

We have shown that if $r(T) \in \sigma(T)$ then

$$\begin{aligned} r(T) &= \inf\{w(\mathcal{R}e(\Delta_\lambda^n(XTX^{-1}))) \mid X \in \mathcal{B}(H) \text{ invertible} \} \\ &= \inf\{\|\mathcal{R}e(\Delta_\lambda^n(XTX^{-1}))\| \mid X \in \mathcal{B}(H) \text{ invertible} \} \end{aligned}$$

Now, if T is an arbitrary operator, then there exists $z \in \sigma(T)$ such that, $|z| = r(T)$. Put $\theta = -\arg(z)$. Then $r(T) = z \exp(i\theta) \in \sigma(\exp(i\theta)T)$. Hence by the first part of de proof, we conclude that

$$\begin{aligned} r(T) = r(\exp(i\theta)T) &= \inf\{w(\mathcal{R}e(\Delta_\lambda^n(\exp(i\theta)XTX^{-1}))) \mid X \in \mathcal{B}(H) \text{ invertible} \} \\ &= \inf\{\|\mathcal{R}e(\Delta_\lambda^n(\exp(i\theta)XTX^{-1}))\| \mid X \in \mathcal{B}(H) \text{ invertible} \}. \end{aligned}$$

This completes the proof of the theorem. \square

As immediate consequence of Theorem 3.2, we obtain the following corollary which is a characterization of normaloid operators

Corollary 3.1. If $T \in \mathcal{B}(H)$, and a natural number n , then the following assertions are equivalent

- (i) T is normaloid;
- (ii) there exists $\theta \in \mathbb{R}$ such that, for all $X \in \mathcal{B}(H)$ invertible

$$\|T\| \leq w(\mathcal{R}e(\Delta_\lambda^n(\exp(i\theta)XTX^{-1})));$$

- (iii) there exists $\theta \in \mathbb{R}$ such that, for all $X \in \mathcal{B}(H)$ invertible

$$\|T\| \leq \|\mathcal{R}e(\Delta_\lambda^n(\exp(i\theta)XTX^{-1}))\|.$$

We end this paper by the following theorem which gives a new formula of the spectral radius of T , in terms of the asymptotic behavior of powers and the numerical radius of T

Theorem 3.3. For every operator $T \in \mathcal{B}(H)$ and for each natural number n , we have

$$r(T) = \lim_k w(\Delta_\lambda^n(T^k))^{1/k}.$$

Proof. For each natural number n and k , we have

$$r(T)^k = r(T^k) = r(\Delta_\lambda^n(T^k)) \leq w(\Delta_\lambda^n(T^k)) \leq \|\Delta_\lambda^n(T^k)\|$$

Hence

$$r(T) \leq w(\Delta_\lambda^n(T^k))^{1/k} \leq \|\Delta_\lambda^n(T^k)\|^{1/k}.$$

By Theorem 2.2, we deduce that

$$r(T) = \lim_k w(\Delta_\lambda^n(T^k))^{1/k}.$$

Which completes the proof. \square

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